Spot bifurcations in three-component reaction-diffusion systems: The onset of propagation

M. Or-Guil,[1,2,*] M. Bode,[1] C. P. Schenk,[1] and H.-G. Purwins[1]
[1]Institut für Angewandte Physik, Universität Münster, Corrensstrasse 2/4, 48149 Münster, Germany[†]
[2]Max Planck Institute for Physics of Complex Systems, Nöthnitzer Strasse 38, 01187 Dresden, Germany
(Received 10 November 1997)

We present an analytical investigation of the bifurcation from stationary to traveling localized solutions in a three-component reaction-diffusion system of arbitrary dimension with one activator and two inhibitors. We show that increasing one of the inhibitors' time constants leads to such a bifurcation. For a limit case, which comprises the full range of stationary two-component patterns, the bifurcation is supercritical and no other bifurcation precedes it. Bifurcation points and velocities close to the branching point are predicted from the shape of the stationary solution. Existence and stability of the traveling solution are checked by means of multiple scales perturbation theory. Numerical simulations agree with the analytical results. [S1063-651X(98)06206-0]

PACS number(s): 82.20.Mj, 47.54.+r, 02.30.−f

I. INTRODUCTION

Stable traveling spots on two-dimensional domains have recently been established as solutions of three-component reaction-diffusion (RD) systems [1,2]. These solutions were obtained by integrating numerically the following set of equations [2]:

$$u_t = D_u(u_{xx}+u_{yy}) + f(u) - v - \kappa_3 w + \kappa_1, \tag{1a}$$

$$\tau v_t = D_v(v_{xx}+v_{yy}) + u - v, \tag{1b}$$

$$\theta w_t = D_w(w_{xx}+w_{yy}) + u - w, \tag{1c}$$

with a cubiclike nonlinear function $f(u)$ and positive parameters τ, θ, κ_3 on a square-shaped domain with periodic boundary conditions. These equations consist of an activator, u, and two inhibiting components, v and w. The inhibitors differ in their time and diffusion length scales; their role in stabilizing a traveling localized solution can be understood as follows: In two-component excitable RD systems [3], as, e.g., in the case θ, D_w, $D_v \to 0$, pulses traveling on one-dimensional domains can be stabilized by the slowly produced and slowly decaying inhibitor v, as this inhibitor couples the dynamics of the leading front to the following back front [4,5]. For an investigation of the stability and the transition from stationary patterns see, for instance, [6] and for the case of traveling stripes in two-dimensional systems see [7]. On higher-dimensional domains, however, the directions perpendicular to the motion of the localized pattern cause difficulties. In these directions, the concept of a back front does not hold and the above mentioned stabilizing mechanism does not apply. As a consequence, the propagating spot usually spreads or shrinks in these directions [2]. To avoid such a decay of the pattern, the second inhibitor, w, was chosen to be fast and strongly diffusing, i.e., θ was chosen to be small and D_w to be large. This provides a smooth, fast reacting distribution of w, which is fast enough to stabilize the shape of the wall of the activator u, see Fig. 1. In contrast to solitons, velocity and shape of the spots approach well defined equilibria which depend on the parameters.

In this article we present an analytical investigation of the bifurcation from stationary to traveling localized solutions and derive an expression for their velocity relating it to the shape of the stationary pattern. Existence and stability of traveling spots are checked by means of multiple scales perturbation theory. Albeit we will focus on highly symmetric patterns [circles in two dimensions (2D), spheres in 3D], our results are more general. In the discussion we will refer to an extension to localized patterns with reduced symmetry and to periodic structures.

II. TRAVELING BIFURCATION

We choose the time scale τ as the bifurcation parameter and start regarding a spot on an infinite domain traveling with the velocity c in the x_1 direction:

$$(\tilde{u},\tilde{v},\tilde{w})(x_1-ct,x_2,\ldots,x_N) = (\tilde{u},\tilde{v},\tilde{w})(z,x_2,\ldots,x_N),$$

where $z := x_1 - ct$ is introduced to describe the spot in a comoving frame. In the framework of Eqs. (1a)–(1c), extended to $N \geq 1$ spatial dimensions, this ansatz leads to

$$-cu_z = D_u\tilde{\Delta}u + f(u) - v - \kappa_3 w + \kappa_1, \tag{2a}$$

$$-c\tau v_z = D_v\tilde{\Delta}v + u - v, \tag{2b}$$

$$-c\theta w_z = D_w\tilde{\Delta}w + u - w, \tag{2c}$$

where $\tilde{\Delta} = \partial_z^2 + \Sigma_{i=2}^N \partial_{x_i}^2$ is the Laplacian in the moving frame. Projection onto $(\tilde{u}_z, -\tilde{v}_z, -\kappa_3\tilde{w}_z)$ leads, after several integrations by parts, to

$$0 = c(\tau\langle\tilde{v}_z^2\rangle + \kappa_3\theta\langle\tilde{w}_z^2\rangle - \langle\tilde{u}_z^2\rangle) =: cS. \tag{3}$$

Here, the brackets denote full spatial integration, and $S = S(c)$ is the abbreviation for the shape- and thus velocity-dependent coefficient. Note that for $\tau = \theta = 0$ c must be zero

*Electronic address: orguil@mpipks-dresden.mpg.de
†URL: http://www.uni-muenster.de/Physik/AP/Purwins

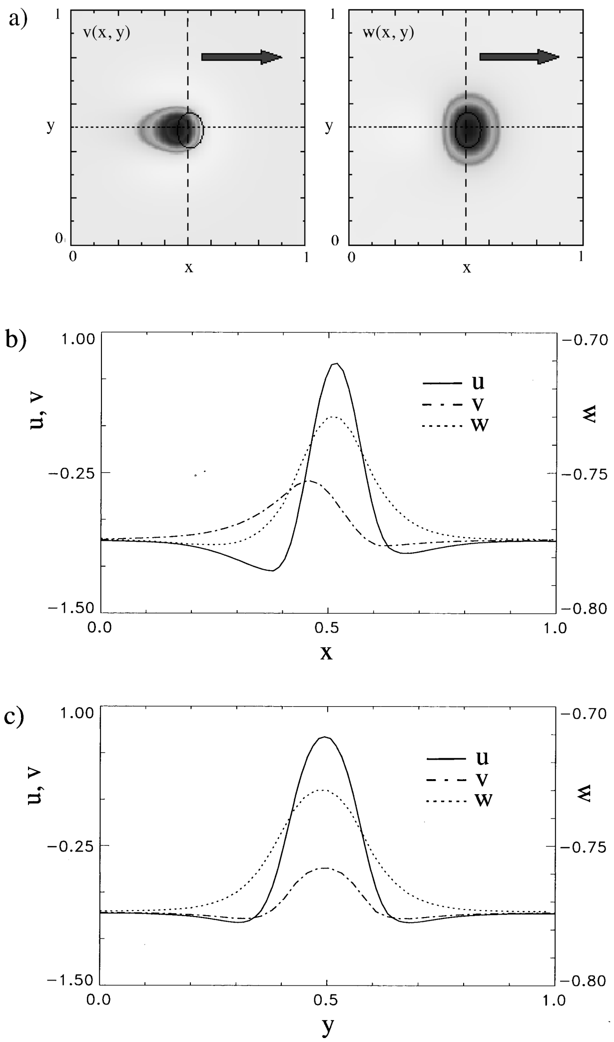

FIG. 1. (a) Traveling spot in two dimensions. The activator u is indicated by the iso-line $u=0$. The inhibitors v and w are shown as gray-scale plots, with dark gray representing high values. Note the shift between the u and the v distribution, which is responsible for the propagation; see also the central cross section (b) in the traveling direction for a more detailed presentation of this shift. (c) Cross section perpendicular to the direction of motion. In this direction the shape of the spot is stabilized by the relatively broad and rapidly reacting second inhibitor field w. Note that, according to theory, this lateral stabilization works in higher dimensions as well. Parameters: $\theta=1$, $D_u=4.67\times10^{-3}$, $D_v=1.25\times10^{-3}$, $D_w=0.064$, $\kappa_1=-6.92$, $\kappa_3=8.5$, and $f(u)=2u-u^3$.

and the solution will not travel. Note also that Eq. (3) allows for stationary solutions $(\overline{u},\overline{v},\overline{w})$, where $c=0$ and S is arbitrary, as well as for traveling ones with $S=0$ and an up to now unspecified velocity c. Thus the bifurcation from stationary to traveling patterns is characterized by $c=S(c=0)=0$. The critical value of the time constant τ follows then immediately from Eq. (3). It is uniquely determined by the shape of the stationary spot, and lies on a straight line in the τ-θ plane:

$$\tau_{\rm crit}=\frac{\langle\overline{u}_x^2\rangle-\kappa_3\theta\langle\overline{w}_x^2\rangle}{\langle\overline{v}_x^2\rangle}. \tag{4}$$

This expression is positive, though, only for a sufficiently fast second inhibitor, i.e., for a small enough value of θ.

Motivated by the stabilizing mechanism discussed in the Introduction, the limit θ, $D_v\rightarrow0$ is of special interest, as this limit provides the fast second inhibitor, w, whereas diffusion of the first one, v, which is not essential, is removed. In this limit, the bifurcation point can be computed explicitly and analytical results concerning the bifurcating branch can be obtained. Making use of Eq. (2b), the shape coefficient S, Eq. (3), can be simplified:

$$S=(\tau-1)\langle\tilde{v}_z^2\rangle-c^2\tau^2\langle\tilde{v}_{zz}^2\rangle.$$

By inspecting this expression, we find that for $\tau<1$ the only way to get $S=0$ is to consider a pattern with $\tilde{v}_z\equiv0$, which, obviously, does not describe a traveling process. Hence, according to Eq. (3), localized patterns do not travel for $\tau<1$. Consequently, the traveling bifurcation has to be supercritical in the limit case we are considering. To make this more explicit, we expand $S=0$ around the point of bifurcation, $\tau_{\rm crit}=1$, $c=0$, in terms of c and $\tau-1$. In lowest order this leads to

$$c^2=(\tau-1)\frac{\langle\overline{v}_x^2\rangle}{\langle\overline{v}_{xx}^2\rangle}, \tag{5}$$

which resembles a pitchfork. This is why, sometimes, it is called a drift pitchfork, especially in one-dimensional systems [6], although linearization of the dynamics around a stationary spot at the bifurcation point yields a $2N$-dimensional null space, and is thus (much) higher degenerate than the generic pitchfork scenario, as will be discussed below. Similar bifurcations have been observed in many different systems [4,6–16]. Note that in the limit case we discuss here, the stationary distributions $\overline{u}$ and $\overline{v}$ are identical, and can therefore be exchanged in Eq. (5).

To check the validity of Eqs. (4) and (5) for traveling spots, we integrated Eqs. (1a)–(1c) with a Crank-Nicholson finite difference scheme on one- and two-dimensional domains, [0,1] and [0,1]×[0,1], applying periodic boundary conditions. To keep CPU time limited we chose $\theta=0.01$. This yields a reasonable agreement with the limit case $\theta=0$, see Fig. 2. Moreover, a deeper investigation in the one-dimensional case showed that extrapolation to $\theta=0$ and an infinitely fine spatial grid results in both the predicted bifurcation point and the predicted slope of $c^2(\tau)$.

So far we described the bifurcation to traveling localized solutions on the level of necessary properties, only. But we did not make any general statement about the existence and the stability of the stationary and the traveling branches. Our numerical results suggest that the traveling bifurcation is the one that destabilizes the previously stable stationary solution when τ is increased. The following section will show that this is indeed the case in the above mentioned limit case. The existence of a stationary spot branch can be mapped to the two-component problem, see the second point of the discussion in Sec. VI. Properties of the traveling branch will be treated in Sec. V.

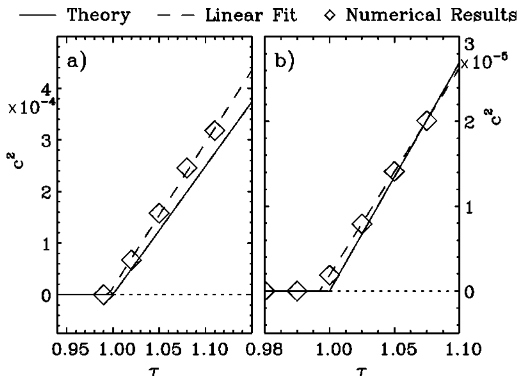

FIG. 2. Onset of propagation. Analytical predictions of the spot velocities (solid lines), see Eq. (5), and numerical results (dashed lines) for (a) the one-dimensional and (b) the two-dimensional case. Parameters: $\theta=0$, $D_u=4.67\times10^{-3}$, $D_v=0$, $D_w=1\times10^{-2}$, $\kappa_3=3.33$, $f(u)=5.67u-u^3$, and (a) $\kappa_1=-0.617$, (b) $\kappa_1=-1.126$. Numerical simulations were performed, though, for $\theta=0.01$ with a spatial grid of (a) 100, (b) 50×50 sites and a step size (a) $\Delta t=1\times10^{-3}$, (b) $\Delta t=6\times10^{-4}$. As predicted by the theory, see Eq. (4), our choice of θ results in a slight shift of the bifurcation. (a) Fit: $\tau_{\text{crit}}=0.9973$; theory: $\tau_{\text{crit}}=0.9969$; (b) fit: $\tau_{\text{crit}}=0.9975$; theory: $\tau_{\text{crit}}=0.9979$.

III. ORDER OF BIFURCATIONS

To find out which one of the possible bifurcations of the steady state comes first, we consider the dynamics linearized around the stationary solution $(\overline{u},\overline{v},\overline{w})$ and compare it with the dynamics in the special case $\tau\to0$. For the limit τ, θ, $D_v\to0$ it can be shown that all the eigenvalues λ are real. For an eigenfunction (ζ,η,ξ) we obtain the eigenvalue problem:

$$\lambda\zeta=D_u\Delta\zeta+f'(\overline{u})\zeta-\eta-\kappa_3\xi,$$

$$0=\zeta-\eta,$$

$$0=D_w\Delta\xi+\zeta-\xi.$$

After removing η this reads

$$\lambda\zeta=D_u\Delta\zeta+f'(\overline{u})\zeta-\zeta-\kappa_3\xi, \tag{6a}$$

$$0=D_w\Delta\xi+\zeta-\xi. \tag{6b}$$

On the other hand, for $\tau>0$, the eigenvalues μ may be complex and we get

$$\mu\zeta=D_u\Delta\zeta+f'(\overline{u})\zeta-\eta-\kappa_3\xi,$$

$$\tau\mu\eta=\zeta-\eta, \tag{7}$$

$$0=D_w\Delta\xi+\zeta-\xi.$$

Again, by removing η and rearranging the equation, this simplifies:

$$\left(\mu-\frac{\tau\mu}{1+\tau\mu}\right)\zeta=D_u\Delta\zeta+f'(\overline{u})\zeta-\zeta-\kappa_3\xi,$$

$$0=D_w\Delta\xi+\zeta-\xi. \tag{8}$$

The operators on the right-hand sides of these equations are the same as in Eqs. (6a) and (6b). Hence, the eigenfunctions, i.e., the components ζ and ξ, are also the same. Comparison of the eigenvalues leads to

$$\lambda=\mu-\frac{\tau\mu}{1+\tau\mu}.$$

This yields for the sought values of μ and their dependence on τ and λ

$$\mu_{1,2}=\frac{\lambda\tau-1+\tau}{2\tau}\pm\sqrt{\frac{\lambda}{\tau}+\left(\frac{\lambda\tau-1+\tau}{2\tau}\right)^2}. \tag{9}$$

In addition, the components $\eta_{1,2}$ of the eigenfunctions read

$$\eta_{1,2}=\frac{1}{1+\mu_{1,2}\tau}\zeta. \tag{10}$$

The conclusions we are looking for can now be drawn from Eq. (9) and from the fact that the λ's are real.

(1) For the traveling bifurcation sought the eigenvalue μ has to be zero. This demands $\lambda=0$, which is understandable since τ is a time constant and thus does not influence the dynamics of a zero eigenfunction.

(2) Bifurcations associated with a given $\lambda<0$ are oscillatory and similar to Hopf bifurcations, though of a higher degeneracy if the rotational symmetry is broken by the eigenfunctions. The respective point of bifurcation is

$$\tau_{\text{bif}}=\frac{1}{1+\lambda}. \tag{11}$$

(3) Equation (11) implies that the bifurcations observed when τ is increased are sorted with respect to the value of the corresponding λ. Hence, if a spot is stable for $\tau\to0$, i.e., if all its λ's are negative besides the N-fold zero eigenvalue corresponding to the translation (Goldstone) modes, this spot will definitely remain stable when τ is increased up to $\tau=1$, where it starts traveling. Any other modes, including breathing perturbations, which as a rule cause the leading destabilization in two-component RD systems [17,18], do not interfere here.

IV. MULTIPLE DEGENERACY

The traveling bifurcation associated with $\lambda=0$, though resembling a pitchfork, is of higher degeneracy. The linearization L,

$$\begin{pmatrix}1&0&0\\0&\tau_{\text{crit}}&0\\0&0&0\end{pmatrix}\begin{pmatrix}\zeta_t\\\eta_t\\\xi_t\end{pmatrix}=T\begin{pmatrix}\zeta_t\\\eta_t\\\xi_t\end{pmatrix}=L\begin{pmatrix}\zeta\\\eta\\\xi\end{pmatrix},$$

is essentially nondiagonal. To understand this situation we will first discuss the one-dimensional case. Consider the two eigenvalues $\mu_1=0$ and $\mu_2=1-1/\tau$ related to the destabilization, Eq. (9). At the point of bifurcation, $\tau_{\text{crit}}=1$, we have $\mu_1=\mu_2=0$. But the corresponding eigenfunctions become equal, too, so there is only one eigenfunction left: the Goldstone mode $g=(\overline{u}_x,\overline{v}_x,\overline{w}_x)^t$. To span the center space we need, and have, a generalized eigenfunction: $e=(0,-\overline{v}_x,0)^t$, which is determined only up to arbitrary additional contributions in the direction of g. Due to the linearized dynamics this mode, which describes a shift of the ''slow'' inhibitor v with respect to the other components, generates the Goldstone mode, i.e., it generates propagation. Hence, at the point of bifurcation, we have $Lg=0$ and $Le=Tg$. Note that T cannot be inverted in the limit case we are interested in. The relevant part of the linear operator L, acting in the (g,e) subspace, reads, after proper normalization,

$$L_{\text{center}}=\frac{1}{\langle \overline{u}_x^2\rangle}T(g,e)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}e^\dagger\\ g^\dagger\end{pmatrix}T.$$

Here, (g,e) and $(e^\dagger,g^\dagger)^t$ are matrices used to transform the representation, with $g^\dagger=(\overline{u}_x,-\overline{v}_x,-\kappa_3\overline{w}_x)$ and $e^\dagger=(\overline{u}_x,0,-\kappa_3\overline{w}_x)$. These vectors obey similar equations as do g and e: $L^\dagger g^\dagger=0$ and $L^\dagger e^\dagger=T^\dagger g^\dagger$, where the operator $T^\dagger$ acts like the matrix $T^t=T$, though from the right. In the N-dimensional case there are N such blocks, corresponding to N Goldstone modes and N generators.

V. NORMAL FORM

To analyze the traveling spot solution branching off from the stationary bough at $\tau=1$ in the limit case D_v, $\theta\to 0$, we start from a given stationary spot (compare the second point of the discussion in Sec. VI) and derive a reduced dynamical system valid for $\tau\approx 1$. Since the velocity of the spot is small close to the bifurcation point, this can be done by means of a multiple time scale perturbation approach as described in the framework of front bifurcations in two-component systems [16].

As discussed in Sec. IV, the relevant degrees of freedom correspond to the Goldstone modes and their generators. Hence, we apply the following ansatz to obtain an approximation globally valid in space:

$$u(r,t)=\overline{u}(r-p)+R_u, \tag{12a}$$

$$v(r,t)=\overline{v}(r-p)+\alpha\cdot\nabla\overline{v}(r-p)+R_v, \tag{12b}$$

$$w(r,t)=\overline{w}(r-p)+R_w, \tag{12c}$$

where we used $r=(x_1,x_2,\ldots)$, the gradient operator $\nabla=(\partial_{x_1},\partial_{x_2},\ldots)$ and time-dependent vectors $p=(p^{(x_1)},p^{(x_2)},\ldots)$ and $\alpha=(\alpha^{(x_1)},\alpha^{(x_2)},\ldots)$ denoting the position of the spot and the amplitudes of the generator modes, respectively. Apart from a slow time scale, it is the generator coefficient α that is introduced in the first order of the perturbation hierarchy. Note the shift of the first inhibitor, v, with respect to the u und w distributions, which is caused by its influence, see Fig. 1. This shift breaks the central symmetry of the stationary spot and enables the motion. Changes of the shape of the spot exceeding the generator part, as they arise from the bifurcation, are captured by the ''rest'' terms $R_{u,v,w}$. These enter the perturbation scheme on the second order level. As slaved degrees of freedom they contribute to the nonlinear terms of the final third order dynamics.

Extending the perturbation procedure from Ref. [16] to the three-component case in a multidimensional situation with D_v, $\theta\to 0$, yields the following third order result:

$$p_t=\alpha, \tag{13a}$$

$$\alpha_t=(\tau-1)\alpha-Q|\alpha|^2\alpha. \tag{13b}$$

The coefficient Q is a positive number to be computed from the shape of the stationary spot solution:

$$Q=\frac{\langle \overline{v}_{xx}^2\rangle}{\langle \overline{v}_x^2\rangle}. \tag{14}$$

The first thing to observe is that there is, indeed, a steadily moving solution to Eq. (13a), and our general result concerning its equilibrium velocity, α^*, Eq. (5), is confirmed:

$$c^2=|\alpha^*|^2=\frac{(\tau-1)}{Q}. \tag{15}$$

To further investigate the stability of the traveling spot it is useful to decompose the generator amplitude α into its absolute value ρ and the normalized vector $\hat{\alpha}$:

$$\alpha=\rho\hat{\alpha}.$$

Since the p dynamics is irrelevant with respect to stability in the present situation, we restrict ourselves to the equations for α assuming that $|\alpha|=\rho\neq 0$. Projection on $\hat{\alpha}$ yields

$$\rho_t=(\tau-1)\rho-Q\rho^3, \tag{16a}$$

$$\hat{\alpha}_t=0. \tag{16b}$$

As a result, we find the absolute value ρ of the velocity vector to be asymptotically stable with an eigenvalue $\tau-1-3Qc^2=-2(\tau-1)$ whereas its direction provides additional neutral degrees of freedom reflecting the isotropy of the original problem. Note that in the case of an N-dimensional domain there are $N-1$ new neutral modes of this type.

VI. DISCUSSION

Up to now, we discussed a three-component extension of a FitzHugh-Nagumo-type reaction-diffusion system which provides stable traveling spots at least on one- and two-dimensional domains, see Fig. 2. This view can be widened.

First, note that there is no need to restrict N to $\{1,2\}$. All the results apply for higher dimensions as well.

Second, the τ, $D_v\to 0$ system can be identified with the following two-component RD system:

$$u_t=D_u\Delta u+f(u)-u-\kappa_3 w+\kappa_1, \tag{17a}$$

$$0=D_w\Delta w+u-w. \tag{17b}$$

This class of equations is known to provide stable stationary localized solutions for suitable values of the parameters. In particular, D_w has to be sufficiently large. Usually, the ratio D_u/D_w is considered asymptotically small to prove this result, see, for instance [17,18], for the existence of stable stationary spots in two and three dimensions (circles, spheres). This is, however, not a completely satisfying limit since many interesting patterns are systematically excluded. For $D_w \gg D_u$, a spot solution is surrounded by a far field decaying (almost) exponentially to the basic homogeneous state. Consider a two-spot state, now. The far fields cause an interaction of the spots. Due to the shape of their fields, and the inhibitory influence of w, this interaction is repulsive in the above case of separated scales. Hence, a stationary two-spot solution does not exist on this length scale. There is an unstable stationary solution, though, with a very small distance between the spots due to the direct interaction of the activator walls. For comparable diffusion coefficients, D_u and D_w, the situation is completely different. The decay of the far field becomes oscillatory with an almost exponential envelope. This implies an oscillation of the interaction with the distance of the spots. The spatial changes from repulsive to attractive interaction lead to (infinitely many) stationary stable moleculelike combinations of the two spots. Through addition of further spots, clusters of almost arbitrary shape can be formed, see [19–21] for one- and two-dimensional systems. Obviously, all these patterns, supplemented by $\overline{v} = \overline{u}$, are also solutions of the three-component system for $D_v \to 0$, and, therefore, obey Eq. (5) and the stability analysis. There is, however, a remark to be made concerning nonsymmetric localized patterns. As opposed to spots, they have additional neutral modes associated with rotations. Hence, according to the linearization, we also expect such a structure to start whirling, and it is up to nonlinear mode interactions to select one of the many possible combinations.

Third, Eqs. (17a) and (17b) are also known to allow periodic stationary solutions such as stripes or hexagons. These patterns are covered as well by our results. Integrations, where needed, have to be performed over suitable finite subdomains to ensure convergence.

Fourth, note that a traveling solution can never be invariant with respect to rotations, since otherwise the direction of propagation would be ambiguous. Therefore any rotational symmetry—either discrete or continuous—of a stationary solution (and reflection symmetry in the one-dimensional case), if there is any, will necessarily be broken by the regarded bifurcation. In this case, the onset of propagation is associated with parity breaking [6,9,12,13]. Symmetry, however, is not a necessary condition for a pattern to be stationary, and, hence, parity breaking is not the actual mechanism causing the bifurcation.

Finally, we want to mention that it was for the sake of simplicity and to make the presentation as explicit as possible that we used FitzHugh-Nagumo-type (FHN) reaction terms throughout the paper. Indeed, the stability analysis can also be performed for the more general system

$$u_t = D_u \Delta u + f(u,w) - v, \tag{18a}$$

$$\tau v_t = u - v, \tag{18b}$$

$$0 = D_w \Delta w + h(u,w). \tag{18c}$$

As above, this yields Eqs. (9)–(11) and their implications. There is no way, though, in general, to write down the vectors $g^\dagger$, e, $e^\dagger$ explicitly.

In a general situation with reaction terms $F(u,v,w)$, $G(u,v,w)$, $H(u,v,w)$ and additional diffusion of component v, the generator modes of a given spot are not known explicitly but can be computed numerically from the linearization around the stationary solution. The same is true for the center-space modes of the adjoint problem which have to be known to perform the projections on the different levels of the perturbation hierarchy.

What kind of reduced dynamical system can we expect on the third order level in such a ''generic'' situation, which is still homogeneous and time invariant, of course? Since an explicit dependence on the position p is impossible, only α defines a distinguished direction, and all the α terms have to be odd to ensure the symmetry $(\alpha, p_t) \to (-\alpha, -p_t)$, there may be at most additional terms with $|\alpha|^2\alpha$ and $(\tau - \tau_{bif})\alpha$ in the p_t equation. However, such terms would always be negligible as compared to the leading α contribution in the same equation. There may be differences, though.

(1) The traveling bifurcation may not be the leading one. In many cases, e.g., a breathing mode comes first as in typical two-component systems [17,18].

(2) The coefficient Q may be negative such that the traveling bifurcation is subcritical, which is also typical in the two-component case [6,7].

(3) The bifurcation will in general be shifted to a value $\tau_{\mathrm{bif}} \neq 1$, of course, see, e.g., Fig. 2. In particular, choosing θ large enough in the framework of the FHN-type model discussed in this article removes the bifurcation completely, see Eq. (4).

ACKNOWLEDGMENTS

The authors thank the HLRZ at the Forschungszentrum Jülich for providing CPU time on the paragon XP/S 10 and the DFG for financial support.

[1] A. N. Zaikin, Fizicheskaya Mysl' Rossii **1**, 54 (1995).
[2] C. P. Schenk, M. Or-Guil, M. Bode, and H.-G. Purwins, Phys. Rev. Lett. **78**, 3781 (1997).
[3] E. Meron, Phys. Rep. **218**, 1 (1992).
[4] J. Rinzel and D. Terman, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **42**, 1111 (1982).
[5] P. Ortoleva and J. Ross, J. Chem. Phys. **63**, 3398 (1975).
[6] M. Kress, L. S. Tuckerman, and D. Barkley, Phys. Rev. A **46**, 5054 (1992).
[7] P. Schütz, M. Bode, and V. V. Gafiichuk, Phys. Rev. E **52**, 4465 (1995).
[8] H. Ikeda, M. Mimura, and Y. Nishiura, Nonlinear Analysis, Theory, Methods & Applications **13**, 507 (1989).
[9] P. Coullet, J. Lega, B. Houchmanzadeh, and J. Lajzerowicz, Phys. Rev. Lett. **65**, 1352 (1990).
[10] A. Hagberg and E. Meron, Nonlinearity **7**, 805 (1994).
[11] C. Elphick, A. Hagberg, and E. Meron, Phys. Rev. E **51**, 3052 (1995).

[12] E. Knobloch, Phys. Rev. Lett. **74**, 4839 (1995).
[13] T. Kawagishi, T. Mizuguchi, and M. Sano, Phys. Rev. Lett. **75**, 3768 (1995).
[14] G. Haas, M. Bär, I. G. Kevrekidis, P. B. Rasmussen, H.-H. Rotermund, and G. Ertl, Phys. Rev. Lett. **75**, 3560 (1995).
[15] M. Bode, A. Reuter, R. Schmeling, and H.-G. Purwins, Phys. Lett. A **185**, 70 (1994).
[16] M. Bode, Physica D **106**, 270 (1997).
[17] T. Ohta, M. Mimura, and R. Kobayashi, Physica D **34**, 115 (1989).
[18] C. B. Muratov and V. V. Osipov, Phys. Rev. E **53**, 3101 (1996).
[19] M. Bode and H.-G. Purwins, Physica D **86**, 53 (1995).
[20] C. P. Schenk, P. Schütz, M. Bode, and H.-G. Purwins, Phys. Rev. E (to be published).
[21] M. Or-Guil, E. Ammelt, F.-J. Niedernostheide, and H.-G. Purwins, in *Nonlinear Dynamics and Pattern Formation in the Natural Environment*, edited by A. Doelman and A. van Harten (Longman, London, 1995).